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Measuring and decomposing firm's revenue and cost efficiency: The Russell measures revisited

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ABSTRACT

Overall inefficiency measurement and decomposition are important for firms facing a world of changing prices since the resultant loss has implications on managers' decision making. In this paper, we draw attention to some problems within existing approaches to decompose overall inefficiency into its sources and propose new revenue and cost inefficiency measures based on the well-known Russell measures. Specifically, the technical inefficiency component is calculated by the Russell output (input) measure, which is able to incorporate all sources of inefficiency corresponding to the output (input) side, specifically output (input) slacks, whereas allocative inefficiency is retrieved residually. All our results are derived from a new Fenchel–Mahler inequality using the theory of convex conjugates. This paper has several implications in theory and practice. From a theoretical point of view, we establish a natural dual relationship between the revenue (cost) function and the Russell output (input) measure; despite the previous unsuccessful attempts in the literature to provide such duality result. From a practical point of view, we provide a way of decomposing revenue (cost) inefficiency into allocative inefficiency and a component that measures technical inefficiency in the sense of Pareto, contrasting with the usual approaches for decomposing revenue (cost) inefficiency, such as those based on Shephard's output (input) distance function and the directional output (input) distance function.

1. Introduction

Data Envelopment Analysis (DEA) is a methodology based fundamentally on mathematical programming for the assessment of technical efficiency of a set of Decision Making Units (DMUs) that use several inputs to produce several outputs. In standard microeconomic theory, the economic behavior of a DMU is usually characterized by cost minimization, revenue maximization, or profit maximization. In particular, if revenue maximization is assumed, the DMU faces exogenously determined market output prices, and we can assume that the objective of each DMU is to choose the output combination that results in the maximum revenue. In this sense, revenue inefficiency measures how close the actual revenue of the evaluated DMU approaches the maximum feasible revenue. Additionally, in the Farrell (1957) tradition[1], revenue inefficiency has usually been decomposed into two components, technical inefficiency and allocative inefficiency, as a way to understand what needs to be done to enhance the performance of the assessed unit.

So far, two famous approaches have traditionally been used in the literature to measure and decompose cost and revenue efficiency. Chronologically, the first one measures revenue (cost) efficiency as the ratio between the optimal revenue (cost) to the actual revenue (cost) corresponding to the assessed DMU, one of the usual economic indexes of a firm's performance, in order to evaluate the level of lost (saved) revenue (cost). In addition, for this first approach, the revenue (cost) efficiency is decomposed into subcomponents using Shephard's output (input) distance function (Shephard, 1953). In particular, the technical efficiency component is calculated as the reciprocal of Shephard's output (input) distance function while allocative efficiency is retrieved as a residual between revenue (cost) and technical efficiency. On the other hand, Shephard's output and input distance functions are closely related to radial measures in DEA, projecting the evaluated unit to the efficient frontier of the technology through equiproportional changes in outputs (inputs) and, consequently, preserving the mix. Unlike Shephard's distance functions, the directional distance function introduced by Chambers et al. (1996, 1998) is a way to make the path to the frontier followed by the assessed DMU more flexible in order to be efficient. In particular, the directional

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[1] As Chambers and Färe (2004, p. 330) pointed out "Since the time of the first frontier analyses (Farrell, 1957), it has been traditional to decompose efficiency (or inefficiency if you are of a pessimistic cast) into two components: technical efficiency and allocative efficiency."

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distance function requires the specification of a directional vector, allowing for a flexible choice of orientation in the input and output space. This notion is also equivalent to Shephard's output distance function when the directional output vector corresponds to the observed output vector for the assessed DMU. Additionally, by duality, it is possible to relate the output directional distance function to the revenue function and establish the corresponding Fenchel–Mahler inequality (Färe and Primont, 2006). In this case, revenue efficiency is estimated as the normalized deviation between optimal revenue and observed revenue for the evaluated unit at market prices, providing a measure of the revenue lost by not operating in a fully efficient manner. The normalization that emerges, in a natural way, from the application of duality theory coincides with the “value” of the directional output vector, i.e., the sum of the product of the components of the directional output vector with their corresponding output prices. Nevertheless, despite the existence of manifold possibilities as directional vectors (Chambers and Färe, 2008), in practice, researchers have considered only a few. For example, one usual selection is the vector corresponding to the observed output vector for the evaluated DMU. However, in this case, the directional distance function is equivalent to Shephard's output distance function, while the approach to measure and decompose revenue efficiency also coincides with the original approach introduced by Farrell (see Färe and Primont, 2006, for more details). Accordingly, this particular output directional distance function suffers the same weakness that we pointed out for Shephard's distance functions associated with the preservation of the mix. On the other hand, another alternative for the directional output vector is to set each vector component equal to one (Färe et al., 2005). However, this option yields an unusual deflator to be used as a way to normalize the revenue efficiency: the normalization would coincide with the sum of output market prices.

In addition, both radial measures and directional distance functions neglect slacks (Ray, 2004). In particular, when the efficiency is measured in an output-oriented context, the evaluated units are projected onto the efficient frontier in the output space and, specifically, output slacks can play an important role in the evaluation of technical efficiency. As mentioned earlier, revenue efficiency has usually been decomposed in the literature into technical efficiency and allocative efficiency. However, one issue that has received little attention in DEA is the decomposition of the gap between optimal and actual revenue (cost) by a technical efficiency measure that takes all types of technical inefficiencies in the corresponding output (input) space into account. Specifically, we are referring, depending on the selected orientation for the analysis, to the output or input slacks of the model. However, the usual approaches to measure revenue (cost) efficiency ignore the possible existence of slacks. Consequently, the corresponding allocative component will be incorrectly estimated as a residual since this component will reflect not only the gains in revenue (cost) that can be accomplished by substitutions along the efficient frontier, but it will also account for some type of technical inefficiency (the slacks).

In view of the preceding discussion, we highlight that the usual approaches for measuring and decomposing revenue and cost efficiency present weaknesses that should be resolved. In this paper, we propose a solution for addressing these problems that is based on the Russell measures (Färe and Lovell, 1978; and Färe et al., 1985). All our results are derived from a new Fenchel–Mahler inequality using the theory of convex conjugates and, consequently, we prove the existence of a new dual correspondence between the revenue (cost) function and the Russell output (input) measure. One advantage to this approach in contrast to others is that it resorts to the Russell measures to estimate the technical inefficiency component, while allocative inefficiency is retrieved residually. It guarantees that, depending on the model's chosen orientation, all sources of technical inefficiency in the corresponding output or input space will be taken into account, since these models are related to the Pareto–Koopmans definition of technical efficiency.

In DEA no measure satisfies all desirable properties for measuring technical efficiency (see, for example, Russell and Schworm, 2009). So, practitioners must select between several “imperfect” alternatives to assess technical efficiency. This point is true if the focus is placed on treating the technical efficiency measures as being completely independent from prices and concepts of economic efficiency. However, Russell (1985) shows that if the existence of a dual relationship with the cost or revenue functions is required as an axiom, then Shephard's distance functions are the adequate selection between all the options since they are the natural dual to the usual measures of economic efficiency, and they are also bounds of the usual measures of economic efficiency, allowing the decomposition of the economic efficiency index. In this case, Russell's reasoning is based on assuming that the Russell measures do not have a dual relationship with the cost or revenue functions. For the same reasons, Kopp (1981, p. 452) claims that “…in terms of the economic concept of cost, $F(u,x)$ seems superior [to the Russell measure]”, where $F(u,x)$ denotes the inverse of Shephard's input distance function. Along the same line, Färe et al. (1985, p. 142), in talking about the Russell measures, claim that “…they do not generate a natural decomposition of overall efficiency.” In contrast to what is believed, we prove in this paper that the Russell output (input) measure of technical efficiency is also a natural dual precursor of the revenue (cost) function. In this respect, a previous attempt to solve this problem was made by Färe et al. (2007), where the original notion is modified in order to define a “multiplicative” version of the Russell measure. To do that, they resorted to the geometric mean as objective function instead of the usual arithmetic mean. In this way, they are able to obtain a decomposition of cost efficiency in terms of the redefined Russell measure, an allocative efficiency component and an unusual third component called “the Debreu–Farrell deviation.” In our case, we resort to the traditional definition of Russell measures, based on the arithmetic mean, and the decomposition that we propose only uses two terms, the Russell measure plus an allocative component.

Recently, Aparicio et al. (2013) introduced a new way to measure and decompose revenue inefficiency. This approach must be highlighted because it also focuses on the issue of taking all sources of technical inefficiency into account when overall inefficiency is decomposed. Therefore, both approaches, that by Aparicio et al. (2013) and the new model we introduce here, are aimed at measuring and decomposing revenue inefficiency by resorting to a technical inefficiency component that takes output slacks into account. However, while the Aparicio et al. (2013) approach is based on Cooper et al. (2011), ours is founded on the well-known Russell output measure of technical efficiency. In this way, and as happens with Shephard's output distance function and the directional output distance functions, the interpretation of the technical efficiency measure used in our case is easier. Specifically, in the case of the Russell output measure, its value can be interpreted as the average of proportional rates of output expansion.

The paper is organized as follows: In Section 2, existing approaches for estimating revenue efficiency are outlined and, additionally, we briefly recall the definition of the Russell measures. Section 3 shows how revenue efficiency can be estimated and decomposed using the Russell measures by applying the theory of convex conjugates. In Section 4, we illustrate the new approach through a numerical example. Section 5 concludes the paper.

2. Preliminary notions

In this section, we briefly review the different procedures that can be found in the literature for measuring and decomposing revenue efficiency. To do this, we hereafter assume that the evaluated units face exogenously determined output prices and try to maximize their revenue. Additionally, in this section a brief description of the Russell measures is provided.

Before briefly reviewing existing literature, let us introduce some notation. Consider that we have observed n Decision Making Units (DMUs) that use m inputs to produce s outputs. These are denoted by (x^j, y^j), $j=1,...,n$. It is assumed $x^j = \left(x_1^j,...,x_m^j\right) \in R_+^m$, $j=1,...,n$, and $y^j = \left(y_1^j,...,y_s^j\right) \in R_{++}^s$, $j=1,...,n$. The relative efficiency of each DMU$_0$ in the sample is assessed with reference to the so-called production possibility set $T = \{(x,y) = (x_1,...,x_m, y_1,...,y_s) : x \text{ can produce } y\}$, which can be empirically constructed from n observations by assuming several postulates (see Banker et al., 1984). If, in particular, variable returns to scale is assumed, then T can be characterized as follows:

$$T = \left\{ (x,y) \in R_{++}^m \times R_{++}^s : \sum_{j=1}^n \lambda_j x_i^j \le x_i, \forall i, \sum_{j=1}^n \lambda_j y_r^j \ge y_r, \forall r, \sum_{j=1}^n \lambda_j = 1, \lambda_j \ge 0, \forall j \right\}. \quad (1)$$

Hereafter, we assume that each DMU is interested in maximizing outputs while using the observed amount of input. This type of approach is called output-oriented in the literature. In order to implement this approach, introducing the output production set is useful. In this sense, for each input vector x, let $P(x)$ be the set of feasible (producible) outputs. Formally, $P(x) = \{y : (x,y) \in T\}$.

Given a fixed level of input $x^0 = (x_1^0,...,x_m^0) \in R_+^m$, let us also define as $r(x^0,p)$ the maximum feasible revenue given the output price vector $p = (p_1,...,p_s) \in R_{++}^s$:

$$r(x^0,p) = \sup_y \left\{ \sum_{r=1}^s p_r y_r : (x^0,y) \in T \right\} = \sup_y \left\{ \sum_{r=1}^s p_r y_r : y \in P(x^0) \right\}. \quad (2)$$

Next, we explicitly show how the value of the revenue function $r(x^0,p)$ can be calculated in DEA as in (3) (see Ray, 2004):

$$\begin{aligned} r(x^0,p) = \quad & \underset{\lambda,y}{Max} \sum_{r=1}^s p_r y_r \\ & s.t. \\ & \sum_{j=1}^n \lambda_j x_i^j \le x_i^0, && i=1,...,m \quad (3.1) \\ & -\sum_{j=1}^n \lambda_j y_r^j + y_r \le 0, && r=1,...,s \quad (3.2) \\ & \sum_{j=1}^n \lambda_j = 1, && (3.3) \\ & \lambda_j \ge 0, && j=1,...,n \quad (3.4) \\ & y_r \ge 0, && r=1,...,s \quad (3.5) \end{aligned} \quad (3)$$

The dual program of (3) is (4)[2].

$$\begin{aligned} & \underset{cd,\psi}{Min} \sum_{i=1}^m c_i x_i^0 + \psi \\ & s.t. \\ & \sum_{i=1}^m c_i x_i{}_j - \sum_{r=1}^s d_r y_r^j + \psi \ge 0, && j=1,...,n \quad (4.1) \\ & d_r \ge p_r, && r=1,...,s (4.2) \\ & c_i \ge 0, && i=1,...,m (4.3) \end{aligned} \quad (4)$$

[2] Really the dual program of model (3) has additionally the non-negativity constraints for the decision variables d_r, $r=1,...,s$. However, as a referee pointed out, this set of constraints is redundant if we consider (4.2) and $p_r > 0$, $r=1,...,s$.

Moreover, we can define the actual revenue for the assessed DMU$_0$, characterized by the input and output vector (x^0, y^0) as $r_0 = \sum_{r=1}^s p_r y_r^0$, where $y_0 = (y_1^0,...,y_s^0) \in R_{++}^s$ denotes the corresponding output vector.

Two different approaches have usually been utilized in the literature to evaluate economic loss due to revenue inefficiency. The first is based on the $r(x^0,p)/r_0$ ratio. This ratio is greater than or equal to one, and is one if and only if the unit being assessed achieves maximum revenue, an interesting property called indication (Portela and Thanassoulis, 2007). Moreover, $r(x^0,p)/r_0$ can be decomposed into a subcomponent of technical efficiency and a subcomponent corresponding to allocative efficiency (AE) as follows: $r(x^0,p)/r_0 = (1/D_o(x^0,y^0)) \cdot AE$, where $D_o(x^0,y^0)$ is the output distance function (Shephard, 1953). The output distance function coincides with the inverse of the output-oriented CCR (Charnes–Cooper–Rhodes) model (see Charnes et al., 1978) and the inverse of the output-oriented BCC (Banker–Charnes–Cooper) model (see Banker et al., 1984) in a DEA framework depending on the assumed returns to scale.

The second approach is based on the directional output distance functions (Chambers et al., 1996, 1998), denoted as $\vec{D}_o(x^0,y^0;g)$, with $g = (g_1,...,g_s) \in R_+^s \setminus \{0_s\}$. In this respect, Färe and Primont (2006) proved that revenue inefficiency may be decomposed into technical inefficiency, $\vec{D}_o(x^0,y^0;g)$, plus allocative inefficiency: $r(x^0,p) - r_0 / \sum_{r=1}^s p_r g_r = \vec{D}_o(x^0,y^0;g) + AI$.

Unfortunately, the two usual approaches for estimating and decomposing revenue efficiency are based on measures that ignore the possible existence of slacks associated with the projected points on the production frontier in order to estimate technical efficiency (see Ray, 2004, p. 95). As a direct consequence, the corresponding allocative efficiency will be incorrectly estimated since this component is measured as a residual related to the difference between the revenue efficiency and technical efficiency measures. In fact, the allocative component will reflect not only the gains in revenue that can be accomplished by substitutions along the efficient frontier in the output space, but it will also account for output slacks.

In this way, the most usual and famous approaches for measuring and decomposing revenue efficiency do not guarantee achieving technical efficiency in the Pareto sense in the output space, i.e., accounting for output slacks, which should be a previous requirement for the measurement of the allocative efficiency as a residual. In this paper, we will try to overcome this drawback by introducing a new approach and showing how it can be used in a numerical example. Next, we concentrate on describing the main characteristics of the Russell measures. In particular, we focus our work on the Russell output measure.

It is well known that technical efficiency measurement started with the works of Debreu (1951), Koopmans (1951), and Shephard (1953). Following them, Farrell (1957) implemented the first measure of technical efficiency. Later, Färe and Lovell (1978) pointed out some limitations with this measure that motivated the development of new approaches in order to measure technical efficiency. In particular, these same authors proposed a new measure, called the Russell input measure of technical efficiency. An output-oriented version of this notion, the Russell output measure of technical efficiency, was similarly defined by Färe et al. (1985). We next show the formulation and main characteristics of this "traditional" Russell output measure of technical efficiency. Nevertheless, we first need to introduce some additional notation.

Given x, the efficient frontier of $P(x)$, also called the weak efficient frontier, is defined as (see Briec, 1998):

$$\partial(P(x)) := \{ y \in P(x) : \hat{y}_r > y_r, r=1,...,s \Rightarrow \hat{y} = (\hat{y}_1,...,\hat{y}_s) \notin P(x) \}. \quad (5)$$

Following Koopmans (1951), in order to measure technical efficiency in the Pareto sense (in other words, taking slacks into account), it is necessary to isolate a particular subset of $\partial(P(x))$. We are referring to the strong efficient frontier, defined below (for more details, see Pastor and Aparicio, 2010):

$$\partial^s(P(x)) := \{y \in P(x) : \hat{y} \geq y, \hat{y} \neq y \Rightarrow \hat{y} \notin P(x)\}. \quad (6)$$

In words, $\partial^s(P(x))$ is the set of all the Pareto–Koopmans efficient points of $P(x)$. Throughout this paper, with the aim of measuring technical efficiency, we will compare the actual performance of each DMU_0, (x^0, y^0), with respect to the points belonging to the strong efficient frontier.

We graphically illustrate both the usual geometry of the output production set and the subsets of its frontier in Fig. 1. The bold solid line corresponds to the strong efficient frontier, consisting of two segments, AB and BC. The weak efficient frontier corresponds to the union between the strong efficient frontier and the dashed lines that appear in the figure.

Now, given a fixed level of input x^0, we turn to the definition of the traditional Russell output measure of technical efficiency (see Färe et al., 1985, p. 149) [3]:

$$R_o(x^0, y^0) = Max\left\{\frac{1}{s}\sum_{r=1}^{s}\phi_r : (\phi_1 y_1^0, \ldots, \phi_s y_s^0) \in P(x^0),\ \phi_r \geq 1, \forall r\right\}. \quad (7)$$

In model (7), ϕ_r evaluates the relative proportional expansion rate of output r, $r = 1, \ldots, s$, whereas the objective function averages these proportional rates of output expansion. Also, in the above formulation, the constraints $\phi_r \geq 1$, $r = 1, \ldots, s$, are the requirements for dominance.

Model (7) can be implemented in DEA as the following linear programming program (see Färe et al., 1985, p. 161):

$$\begin{array}{llll} R_o(x^0, y^0) = & \underset{\phi,\lambda}{Max} & \frac{1}{s}\sum_{r=1}^{s}\phi_r & \\ & s.t. & & \\ & & \sum_{j=1}^{n}\lambda_j x_i^j \leq x_i^0, \quad i = 1, \ldots, m & (8.1) \\ & & -\sum_{j=1}^{n}\lambda_j y_r^j + \phi_r y_r^0 \leq 0, \quad r = 1, \ldots, s & (8.2) \\ & & \sum_{j=1}^{n}\lambda_j = 1, & (8.3) \\ & & -\phi_r \leq -1, \quad r = 1, \ldots, s & (8.4) \\ & & \lambda_j \geq 0, \quad j = 1, \ldots, n & (8.5) \end{array} \quad (8)$$

The value of $R_o(x^0, y^0)$ can be equivalently obtained from the dual program of (8):

$$\begin{array}{llll} R_o(x^0, y^0) = & \underset{\nu,\mu,\alpha,\tau}{Min} & \sum_{i=1}^{m}\nu_i x_i^0 + \alpha - \sum_{r=1}^{s}\tau_r & \\ & s.t. & & \\ & & \sum_{i=1}^{m}\nu_i x_i^j - \sum_{r=1}^{s}\mu_r y_r^j + \alpha \geq 0, \quad j = 1, \ldots, n & (9.1) \\ & & \mu_r y_r^0 - \tau_r = \frac{1}{s}, \quad r = 1, \ldots, s & (9.2) \\ & & \nu_i \geq 0, \quad i = 1, \ldots, m & (9.3) \\ & & \mu_r \geq 0, \quad r = 1, \ldots, s & (9.4) \\ & & \tau_r \geq 0, \quad r = 1, \ldots, s & (9.5) \end{array} \quad (9)$$

[3] The definition of the Russell output measure that we use in this paper does not allow any output of the assessed unit to be zero. Otherwise, it can be proven that the linear program associated with this measure would be unbounded. Nevertheless, the Russell output measure could be modified by deleting the decision variables related to the expansion rate of each output with a zero value for the evaluated unit. It can also be proven that, in this case, our results on duality remain true as long as we slightly modify the revenue function. All these steps and results are available from the authors by request.

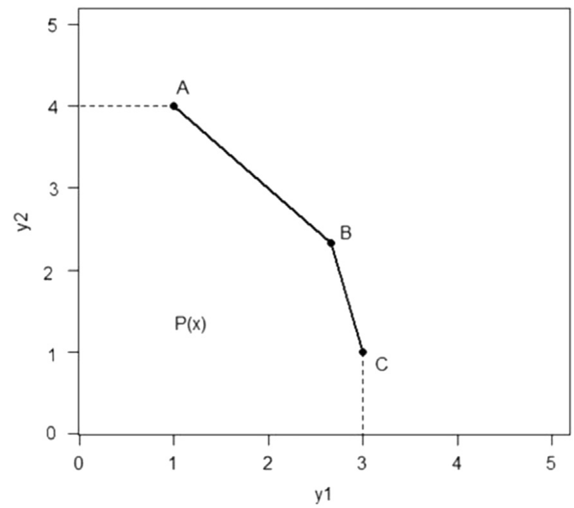

Fig. 1. Illustration of the weak and strong efficient frontier.

As previously mentioned, the efficiency assessment of each DMU_0 is generally obtained as the result of its comparison with a dominating projection point on the efficient frontier of the output production set. The coordinates of this projection will be the targets for DMU_0. Regarding the Russell output measure of technical efficiency, given an optimal solution of (8), (ϕ^*, λ^*), the targets are defined as $y_r^{0*} = \sum_{j=1}^{n}\lambda_j^* y_r^j$, $r = 1, \ldots, s$. The following proposition establishes that, unlike other approaches, such as Shephard's output distance function and the directional output distance function, the targets generated from (8) are always Pareto–Koopmans efficient points of $P(x^0)$.

Proposition 1. (Färe et al., 1985). *Let (ϕ^*, λ^*) be an optimal solution of (8). Then $y^{0*} = (y_1^{0*}, \ldots, y_s^{0*}) \in \partial^s(P(x^0))$.*

Before continuing, we think it is interesting to show the properties that the traditional Russell output measure of technical efficiency satisfies. With respect to this point, Färe and Lovell (1978) were the first to propose a set of desirable properties that an ideal efficiency measure should meet. Later, Cooper et al. (1999) and Pastor et al. (1999) listed similar requirements for the DEA context and suggested some others. In particular, the main properties are (P1), the measure should be greater than or equal to one, with one meaning full-efficiency; (P2), the assessed DMU_0 is Pareto–Koopmans efficient if and only if the measure takes a value of one; (P3), units invariant; and finally, (P4), strong monotonicity. Specifically, for the output-oriented case, strong monotonicity means that by maintaining all other inputs and outputs constant, an increase in any of its outputs will increase the efficiency score for an inefficient DMU_0.

Proposition 2. (Färe et al., 1985*; and* Pastor et al., 1999). *Let (x^0, y^0) be the input and output vector corresponding to DMU_0. Then, the following is true for the Russell output measure:*

(i). $R_o(x^0, y^0) \geq 1$.
(ii). $R_o(x^0, y^0) = 1$ if and only if $y^0 \in \partial^s(P(x^0))$.
(iii). $R_o(x^0, y^0)$ is units invariant.
(iv). $R_o(x^0, y^0)$ is strongly monotonic in outputs.

The Russell output measure of technical inefficiency is associated with the Russell output measure of technical efficiency, i.e.,

$$RI_o(x^0, y^0) = R_o(x^0, y^0) - 1. \quad (10)$$

This measure satisfies the following characteristics: It is always greater than or equal to zero, with zero signaling full-efficiency; it takes a value of zero if and only if the evaluated unit is Pareto–Koopmans efficient; it is also units invariant and strongly monotonic.

There is an important stream in the literature that has criticized the Russell measures, e.g., Russell (1985, 1988, 1990), Bol (1986), and Dmitruk and Koshevoy (1991), whereas other authors have promoted the use of this approach, such as Färe and Lovell (1978), Färe et al. (1983), and Färe et al. (1985). The pointed weaknesses of the Russell measures are based on the properties that these measures do not satisfy. Nevertheless, more recently, and working on data-generated technologies (e.g., DEA), Russell and Schworm (2009) evaluated the Debreu–Farrell measure (the inverse of Shephard's distance function), the Russell measures, and the Zieschang index with respect to indication, monotonicity, homogeneity, and continuity. Their main results state that (1) restriction to DEA adds continuity to the set of properties satisfied by the Russell measures (thus eliminating a relevant advantage of the Debreu–Farrell measure), (2) none of the three analyzed efficiency measures satisfy all the axioms and, therefore, (3) trade-offs among the indexes remain. In particular, under DEA, the Russell measures satisfy indication, strict monotonicity (if the origin is excluded from the technology), and continuity, whereas the Debreu–Farrell measure satisfies homogeneity and continuity. Another related work is that by Fukuyama et al. (2014), where a good revision of this type of literature may be found.

In view of the preceding results, the choice between the Debreu–Farrell index and the Russell measures for DEA technologies depends on the importance of homogeneity in technologies versus indication and monotonicity. Hence, the selection between these measures depends upon the investigator's opinion about the relative attractiveness of the existing traditional axioms (Russell and Schworm, 2009). In other words, under a DEA framework, no measure satisfies all desirable properties, so we must choose between several "imperfect" alternatives in practice to assess technical efficiency, one of them being the Russell measure that we use. We specifically select this approach in this paper because assuring Pareto–Koopmans efficiency is really important in our context.

3. Measuring and decomposing revenue efficiency through the Russell output measure

Duality theory has acquired great popularity in microeconomics (see, for example, Varian, 1992; Cornes, 1992; Färe and Primont, 1995; and Luenberger, 1995). Duality theory allows stating the most common alternative ways of representing preferences and technologies, such as indirect utility and expenditure functions, revenue and distance functions, and so on. In general, having different ways to describe a technology seems very useful since some types of mathematical arguments are easier to demonstrate by using, for example, a revenue function instead of a distance function, which is, a direct representation of the technology (see Varian, 1992, p. 81). Both the revenue function and the distance function are, by definition, optimization problems. Specifically, duality theory studies under which conditions these two optimization problems are related.

As we mentioned in the previous section, under a DEA framework, no measure among those existing in the literature satisfies all desirable properties. In this way, researchers must select between several imperfect alternatives in practice to assess technical efficiency. However, this point is true if the focus is placed on treating the technical efficiency measures as being completely independent from prices and concepts of economic efficiency (Russell, 1985). Russell (1985) shows that if the existence of a dual relationship with the cost or revenue functions is required as an axiom (along the same line as indication, monotonicity, and homogeneity), then the Debreu–Farrell measure is the adequate selection between all the options since: (1) It is the natural (quantity based) dual to the usual measure of economic efficiency, and (2) it is a bound of the usual measure of economic efficiency (using market prices), which permits decomposing the economic efficiency index. In this section, we specifically prove that the Russell output measure of technical efficiency is also a dual precursor of the revenue function and, therefore, it shares this good property with the Debreu–Farrell measure.

Particularly, this section is devoted to proving the existing duality relationship between the Russell output measure and the revenue function. In this respect, we prove that the Russell output measure of inefficiency in DEA can be seen as the (minus) Fenchel conjugate of the revenue function given a specific level of inputs. This result is completely new in DEA literature. So far, what is well known is that both Shephard's output distance function and the directional output distance function have dual relationships with the firm's revenue function. Thus, we understand that if the directional output distance function enhanced the approach proposed by Farrell and Shephard, allowing more flexibility in the selected direction of the projection on the efficient frontier, the Russell output measure permits taking the output slacks into account when technical efficiency is measured on the output side, improving the existing approaches in some sense. Consequently, the duality results we show in this section, corresponding to the Russell output measure and the revenue function, could be of interest for DEA researchers and practitioners. Additionally, by applying the well-known Young–Fenchel inequality, we will be able to show in this section how revenue inefficiency can be decomposed into its usual components: technical inefficiency and allocative inefficiency.

Before proving the main theorem of this contribution, let us introduce one definition and two results associated with the standard theory of convex conjugates (see Rockafellar, 1970, pp. 102–106; and Zalinescu, 2002, pp. 75–77).

Definition 1. Let $f: Z \to R$ be a convex function. Then its Fenchel conjugate function, f^*, is defined as $f^*(z^*) := \sup\{\langle z, z^* \rangle - f(z) : z \in Z\} = -\inf\{f(z) - \langle z, z^* \rangle : z \in Z\}$.

Proposition 3 (The Young–Fenchel inequality).. *Let $f: Z \to R$ be a convex function and let f^* be its associated Fenchel conjugate, then $f(z) + f^*(z^*) \geq \langle z, z^* \rangle,\ \forall z \in Z,\ \forall z^* \in Z^*$.*

Proposition 4. *Let f^* be the Fenchel conjugate function of f. Then, the Fenchel conjugate function of f^* is $f^{**} = f$.*

We now focus the analysis on the revenue function. Given the technology T and the input level x^0, the revenue function $r(x^0, p)$ in (2) is a convex function in p (see Färe and Primont, 1995). For DMU$_0$, with vector (x^0, y^0), let us now consider the set $Z^0 = \{p \in R^s : \min\{s y_1^0 p_1, \ldots, s y_s^0 p_s\} \geq 1\}$. Then, applying Definition 1, the Fenchel conjugate function of $r(x^0, p)$ restricted to prices that belong to Z^0 would be

$$r^*(x^0, y^0) = \sup_p \left\{ \sum_{r=1}^{s} p_r y_r^0 - r(x^0, p) : p \in Z^0 \right\}$$
$$= -\inf_p \left\{ r(x^0, p) - \sum_{r=1}^{s} p_r y_r^0 : p \in Z^0 \right\}. \qquad (11)$$

We next prove that the Russell output measure of technical inefficiency $RI_o(x^0, y^0)$ can be written as $\inf_p \{ r(x^0, p) - \sum_{r=1}^{s} p_r y_r^0 : p \in Z^0 \}$ and, therefore, this inefficiency measure coincides with the minus Fenchel conjugate of the revenue function. This result will later allow us to decompose the revenue inefficiency of DMU$_0$ into technical inefficiency and allocative inefficiency for any observed market price p, regardless of whether p belongs to Z^0 or not. Nevertheless, before stating this result, we need to prove a lemma that will be used later.

Lemma 1. *Let* (ν,μ,α,τ) *be a feasible solution of (9). Then,*

$$r(x^0,\mu)-\sum_{r=1}^{s}\mu_r y_r^0 \le \sum_{i=1}^{m}\nu_i x_i^0+\alpha-\sum_{r=1}^{s}\mu_r y_r^0. \tag{12}$$

Proof. Using constraint (9.1) in (9), we have that $\alpha \ge \sum_{r=1}^{s}\mu_r y_r^j-\sum_{i=1}^{m}\nu_i x_i^j$ for all $j=1,\dots,n$, which is equivalent to

$$\alpha \ge \max_{j=1,\dots,n}\left\{\sum_{r=1}^{s}\mu_r y_r^j-\sum_{i=1}^{m}\nu_i x_i^j\right\}. \tag{13}$$

On the other hand, given input and output prices (w,p), Ray (2007),p. 233 showed that, under VRS, the profit function[4] calculated on T corresponding to these prices can be easily obtained as $\pi(w,p)=\max_{j=1,\dots,n}\left\{\sum_{r=1}^{s}p_r y_r^j-\sum_{i=1}^{m}w_i x_i^j\right\}$. This implies that $\alpha\ge\pi(\nu,\mu)$ (using (13)). Additionally, Färe and Primont (1995),p. 136 showed that $\pi(w,p)=\sup_x\left\{r(x,p)-\sum_{i=1}^{m}w_i x_i\right\}$. In this way, we have that $\alpha\ge\pi(\nu,\mu)=\sup_x\left\{r(x,\mu)-\sum_{i=1}^{m}\nu_i x_i\right\}\ge r(x^0,\mu)-\sum_{i=1}^{m}\nu_i x_i^0$, where the last inequality is true because $\sup_x\left\{r(x,\mu)-\sum_{i=1}^{m}\nu_i x_i\right\}$ is always greater than or equal to $r(x,\mu)-\sum_{i=1}^{m}\nu_i x_i$ evaluated at the particular point $x=x^0$. In this fashion, we have obtained that $\alpha\ge r(x^0,\mu)-\sum_{i=1}^{m}\nu_i x_i^0$, which is equivalent to $r(x^0,\mu)\le\sum_{i=1}^{m}\nu_i x_i^0+\alpha$ by rearranging terms. Finally, by adding the term $\left(-\sum_{r=1}^{s}\mu_r y_r^0\right)$ to both sides of the inequality, we get (12).

Now we are ready to prove the desired result: $RI_o(x^0,y^0)=-r^*(x^0,y^0)$. In order to prove the equality, we separately prove the two corresponding inequalities.

Proposition 5. $RI_o(x^0,y^0)\ge -r^*(x^0,y^0)$.

Proof. On the one hand, seeking simplicity, let us denote as $F_{(9)}$ the feasible set of program (9). By (9), $R_o(x^0,y^0)=\inf_{\nu,\mu,\alpha,\tau}\left\{\sum_{i=1}^{m}\nu_i x_i^0+\alpha-\sum_{r=1}^{s}\tau_r:(\nu,\mu,\alpha,\tau)\in F_{(9)}\right\}$. Now, we focus on the term $\sum_{r=1}^{s}\tau_r$. By constraint (9.2) in (9), we have that $\tau_r=\mu_r y_r^0-\frac{1}{s}$, $r=1,\dots,s$, which implies that $\sum_{r=1}^{s}\tau_r=\sum_{r=1}^{s}\mu_r y_r^0-1$. Replacing $\sum_{r=1}^{s}\tau_r$ by $\sum_{r=1}^{s}\mu_r y_r^0-1$, we then have that

$$R_o(x^0,y^0)=\inf_{\nu,\mu,\alpha,\tau}\left\{\sum_{i=1}^{m}\nu_i x_i^0+\alpha-\sum_{r=1}^{s}\mu_r y_r^0+1:(\nu,\mu,\alpha,\tau)\in F_{(9)}\right\}\underbrace{=}_{\substack{\text{Taking}+1\text{ out}\\ \text{of the infimum}}}$$

$$\inf_{\nu,\mu,\alpha,\tau}\left\{\sum_{i=1}^{m}\nu_i x_i^0+\alpha-\sum_{r=1}^{s}\mu_r y_r^0:(\nu,\mu,\alpha,\tau)\in F_{(9)}\right\}+1\underbrace{\Leftrightarrow}_{\substack{\text{Rearranging}\\ \text{terms}}}$$

$$\underbrace{RI_o(x^0,y^0)}_{\substack{=R_o(x^0,y^0)-1\\ \text{(by (10))}}}=\inf_{\nu,\mu,\alpha,\tau}\left\{\sum_{i=1}^{m}\nu_i x_i^0+\alpha-\sum_{r=1}^{s}\mu_r y_r^0:(\nu,\mu,\alpha,\tau)\in F_{(9)}\right\}. \tag{14}$$

On the other hand, by Lemma 1, $r(x^0,\mu)-\sum_{r=1}^{s}\mu_r y_r^0\le\sum_{i=1}^{m}\nu_i x_i^0+\alpha-\sum_{r=1}^{s}\mu_r y_r^0$ for all $(\nu,\mu,\alpha,\tau)\in F_{(9)}$. Then, taking the infimum on both sides of the inequality, we have that

$$\inf_{\nu,\mu,\alpha,\tau}\left\{r(x^0,\mu)-\sum_{r=1}^{s}\mu_r y_r^0:(\nu,\mu,\alpha,\tau)\in F_{(9)}\right\}\le$$

$$\inf_{\nu,\mu,\alpha,\tau}\left\{\sum_{i=1}^{m}\nu_i x_i^0+\alpha-\sum_{r=1}^{s}\mu_r y_r^0:(\nu,\mu,\alpha,\tau)\in F_{(9)}\right\}\underbrace{=}_{\text{by (14)}}RI_o(x^0,y^0). \tag{15}$$

In this way, we have derived the sign of the inequality we were seeking. To finish the proof, we need to show that $-r^*(x^0,y^0)\le(15)$ since, as we have just shown, $(15)\le RI_o(x^0,y^0)$. To do that, note that the infimum in (15) takes a value greater than or equal to the same infimum calculated on a set that contains $F_{(9)}$:

$$\inf_{\nu,\mu,\alpha,\tau}\left\{r(x^0,\mu)-\sum_{r=1}^{s}\mu_r y_r^0:\ \min\{sy_1^0\mu_1,\dots,sy_s^0\mu_s\}\ge 1\right\}\le$$

$$\inf_{\nu,\mu,\alpha,\tau}\left\{r(x^0,\mu)-\sum_{r=1}^{s}\mu_r y_r^0:(\nu,\mu,\alpha,\tau)\in F_{(9)}\right\}. \tag{16}$$

This last inequality is true because, by (9.2) and (9.5) in (9), any $(\nu,\mu,\alpha,\tau)\in F_{(9)}$ satisfies $y_r^0\mu_r\ge\frac{1}{s}$, $r=1,\dots,s\Leftrightarrow sy_r^0\mu_r\ge 1$, $r=1,\dots,s\Leftrightarrow\min\{sy_1^0\mu_1,\dots,sy_s^0\mu_s\}\ge 1$. Finally, (16) is equivalent to $\inf_\mu\left\{r(x^0,\mu)-\sum_{r=1}^{s}\mu_r y_r^0:\ \min\{sy_1^0\mu_1,\dots,sy_s^0\mu_s\}\ge 1\right\}$ since the only decision variable that appears in (16) is μ and, additionally, $\inf_\mu\left\{r(x^0,\mu)-\sum_{r=1}^{s}\mu_r y_r^0:\ \min\{sy_1^0\mu_1,\dots,sy_s^0\mu_s\}\ge 1\right\}=\inf_\mu\left\{r(x^0,\mu)-\sum_{r=1}^{s}\mu_r y_r^0:\mu\in Z^0\right\}=-r^*(x^0,y^0)$ (by (11)). This last step concludes the proof.

Proposition 6. $RI_o(x^0,y^0)\le -r^*(x^0,y^0)$.

Proof. Let $p\in R_{++}^s$ and let (c^*,d^*,ψ^*) be an optimal solution of model (4). Then,

$$r(x^0,p)-\sum_{r=1}^{s}p_r y_r^0=\left(\sum_{i=1}^{m}c_i^* x_i^0+\psi^*\right)-\sum_{r=1}^{s}p_r y_r^0, \tag{17}$$

[4] The profit function is usually defined in the literature as $\pi(w,p)=\sup_{x,y}\left\{\sum_{r=1}^{s}p_r y_r-\sum_{i=1}^{m}w_i x_i:(x,y)\in T\right\}$ (see Färe and Primont, 1995, p. 125).

since, by duality in linear programming and using (3) and (4), $r(x^0,p)=\sum_{i=1}^{m}c_i^*x_i^0+\psi^*$. In addition,

$$\sum_{i=1}^{m}c_i^*x_i^j-\sum_{r=1}^{s}p_ry_r^j+\psi^*\underbrace{\geq}_{\text{by (4.2)}}\sum_{i=1}^{m}c_i^*x_i^j-\sum_{r=1}^{s}d_r^*y_r^j+\psi^*\underbrace{\geq}_{\text{by (4.1)}}0. \quad (18)$$

In this way, (c^*,p,ψ^*,δ) with $p\in Z^0$ and $\delta_r:=p_ry_r^0-(1/s)$, $r=1,...,s$, is a feasible solution of (9) since (9.1) is satisfied by (18), (9.2) by the definition of δ_r, (9.3) by (4.3), (9.4) on the assumption that $p\in Z^0\subset R^s_{++}$ and, finally, (9.5) holds thanks to $p\in Z^0=\{p\in R^s:\ \min\{sy_1^0p_1,...,sy_s^0p_s\}\geq 1\}=\{p\in R^s: p_ry_r^0\geq 1/s, r=1,...,s\}$ and, therefore, $\delta_r=p_ry_r^0-(1/s)\geq 0$ for all $r=1,...,s$. Continuing with the proof, $R_o(x^0,y^0)\leq\sum_{i=1}^{m}c_i^*x_i^0+\psi^*-\sum_{r=1}^{s}p_ry_r^0+1$ by evaluating the objective function of (9) at (c^*,p,ψ^*,δ). Therefore, $R_o(x^0,y^0)$ is a lower bound of $\left\{\sum_{i=1}^{m}c_i^*x_i^0+\psi^*-\sum_{r=1}^{s}p_ry_r^0+1: p\in Z^0\right\}$ and

$$R_o(x^0,y^0)\leq\inf_p\left\{\sum_{i=1}^{m}c_i^*x_i^0+\psi^*-\sum_{r=1}^{s}p_ry_r^0+1: p\in Z^0\right\}, \quad (19)$$

since the infimum of a set is the greatest lower bound of the set.

Now, using (15) we replace $\left(\sum_{i=1}^{m}c_i^*x_i^0+\psi^*-\sum_{r=1}^{s}p_ry_r^0\right)$ by $\left(r(x^0,p)-\sum_{r=1}^{s}p_ry_r^0\right)$ in (19), obtaining $R_o(x^0,y^0)\leq\inf_p\{r(x^0,\mu)-\sum_{r=1}^{s}\mu_ry_r^0+1: p\in Z^0\}$. Taking $+1$ out of the infimum, we have that $R_o(x^0,y^0)\leq\inf_p\left\{r(x^0,\mu)-\sum_{r=1}^{s}\mu_ry_r^0: p\in Z^0\right\}+1$. Finally, by rearranging terms in the inequality and using (10), we have that $RI_o(x^0,y^0)\leq\inf_p\left\{r(x^0,p)-\sum_{r=1}^{s}p_ry_r^0: p\in Z^0\right\}\underbrace{=}_{\text{by (11)}}-r^*(x^0,y^0)$.

Finally, and thanks to Propositions 5 and 6, we have the following result.

Theorem 1. $RI_o(x^0,y^0)=-r^*(x^0,y^0)$.

We have just shown that the Russell output measure of technical inefficiency is dual to the firm's revenue function. Specifically, $RI_o(x^0,y^0)=-r^*(x^0,y^0)=\inf_p\left\{r(x^0,p)-\sum_{r=1}^{s}p_ry_r^0: p\in Z^0\right\}$. In other words, this result says that if we start with the revenue function, we can then derive the technical inefficiency associated with the Russell output measure by minimizing $r(x^0,p)-\sum_{r=1}^{s}p_ry_r^0$ over Z^0. So, the question at this moment is whether the inverse result holds, i.e., whether it is possible to derive the revenue function from the knowledge of the Russell output measure. By Proposition 4, we know that the answer to this question is affirmative because the conjugate r^{**} of r^* is the original r. In this way, we have that

$$r(x^0,p)=\sup_y\left\{\sum_{r=1}^{s}p_ry_r-r^*(x^0,y): y\in Z^{0*}\right\}$$
$$=\sup_y\left\{\sum_{r=1}^{s}p_ry_r+RI_o(x^0,y): y\in R^s_{++}\right\}, \quad (20)$$

since $Z^{0*}=R^s_{++}$ is the domain of r^{**}. This result means that we can recover the revenue function from the Russell output measure. Therefore, we point out that these types of dual relationships make it easy to pass back and forth between a revenue approach and a Russell measure approach when the analysis of the firm's efficiency is the focus.

We now turn to the decomposition of the revenue inefficiency in terms of technical inefficiency and allocative inefficiency. To do so, we go on to apply the well-known Young–Fenchel inequality (Proposition 3) in our context.

Corollary 1. $\underbrace{r(x^0,p)-\sum_{r=1}^{s}p_ry_r^0}_{\text{RevenueInefficiency}}\geq\underbrace{RI_o(x^0,y^0)}_{\text{TechnicalInefficiency}}$ *for all* $p\in R^s_+$ *and all* $y^0\in R^s_{++}$ *such that* $\min\{sy_1^0p_1,...,sy_s^0p_s\}\geq 1$.

Proof. By Theorem 1, $RI_o(x^0,y^0)=-r^*(x^0,y^0)$ or, equivalently, $-RI_o(x^0,y^0)=r^*(x^0,y^0)$. By adding $r(x^0,p)$ to the both sides of the last equation, we have that $r(x^0,p)-RI_o(x^0,y^0)=r(x^0,p)+r^*(x^0,y^0)$, which by Proposition 3 is greater than or equal to $\sum_{r=1}^{s}p_ry_r^0$ for all $p\in R^s_+$ and all $y^0\in R^s_{++}$ such that $\min\{sy_1^0p_1,...,sy_s^0p_s\}\geq 1$. Finally, by rearranging terms we get that $r(x^0,p)-\sum_{r=1}^{s}p_ry_r^0\geq\underbrace{RI_o(x^0,y^0)}_{\text{TechnicalInefficiency}}$.

Relationships like that shown in Corollary 1 allow the decomposition of revenue inefficiency into its usual subcomponents. The technical inefficiency term appeared in a natural way as a direct consequence of the theory of Fenchel conjugates, while the other component, the allocative inefficiency, will emerge as a residual following Farrell's tradition. Before closing the inequality, we would like to highlight one interesting point. The above inequality is only valid for market output prices so that $\min\{sy_1^0p_1,...,sy_s^0p_s\}\geq 1$. This seems to be a weakness in contrast to the usual Fenchel–Mahler inequalities in the literature. Nevertheless, we are able to derive a new Fenchel–Mahler inequality, satisfied by any vector of market output prices. In particular, this result is possible thanks to one property of the revenue function: homogeneity of degree $+1$ in prices (see Färe and Primont, 1995, p. 49). Let us show how. Let $p\in R^s_{++}$. Then, $(p/\min\{sy_1^0p_1,...,sy_s^0p_s\})\in Z^0$. Now, by Corollary 1 we have that

$$r(x^0,p/\min\{sy_1^0p_1,...,sy_s^0p_s\})-\frac{\sum_{r=1}^{s}p_ry_r^0}{\min\{sy_1^0p_1,...,sy_s^0p_s\}}\geq RI_o(x^0,y^0). \quad (21)$$

Finally, by the homogeneity of degree $+1$ of the revenue function, we obtain

$$\underbrace{\frac{r(x^0,p)-\sum_{r=1}^{s}p_ry_r^0}{s\cdot\min\{p_1y_1^0,...,p_sy_s^0\}}}_{\text{NormalizedRevenueInefficiency(NRI)}}\geq\underbrace{RI_o(x^0,y^0)}_{\text{TechnicalInefficiency(TI)}}. \quad (22)$$

The expression on the left of (22), NRI, provides a measure of the firm's revenue lost by not operating in a fully efficient manner. NRI is really a normalized deviation between optimal revenue, $r(x^0,p)$, and observed revenue, $\sum_{r=1}^{s}p_ry_r^0$, at market prices, p. In particular, the normalization that emerges in a natural way from the duality theory is s, the number of outputs multiplied by the minimum among the partial actual revenues corresponding to each output. Thanks to the normalization, it is easy to prove that NRI satisfies a desirable index number property: it is homogeneous of degree zero in prices, which makes NRI invariant to the currency units for the output prices. As Nerlove (1965, p. 94) pointed out, a non-normalized measure like the NRI numerator could be considered as being an inappropriate economic measure due to its homogeneity of degree one in prices. As such, a measure of this type, which in turn is used as an index number, would be inaccurately expressed in nominal money units. Thus, normalization is always needed for these kinds of indexes.

Regarding the properties that NRI should satisfy, Kuosmanen et al. (2010, pp. 585–586) suggested that "the measure [should be] homogenous of degree zero in prices and quantities". As aforementioned, NRI is homogeneous of degree zero in prices. As for homogeneity of degree zero in quantities (outputs), we are going to show that NRI meets this property. For the sake of illustration, let us suppose that we have only one input (x), labor, measured in hours and only one output (y), corn, measured in tons. Let us also suppose that p=\$3000 per ton. Now imagine that we want to change the units of measurement of the output for unit 0, y^0. We particularly want to measure the output in kilograms ($y^{0,new}=1000y^0$). This change in the units of measurement, working in kilograms instead of tons, implies a transformation in p. Indeed, the new p value, p^{new}, is \$3 (3000/1000) per kilogram. It is easy to realize that both $r(x^0, p^{new})$ and $p^{new}y^{0,new}$, measured in \$, take the same original value. Finally,

$$NRI^{new}=\frac{r(x^0,p^{new})-p^{new}y^{0,new}}{p^{new}y^{0,new}}=\frac{r(x^0,p)-py^0}{py^0}=NRI. \tag{23}$$

In order to finish this section, we next show how to calculate allocative inefficiency from (22). Inequality (22) can be turned into equality by including a residual component measuring allocative inefficiency: NRI=TI+AI[5], where

$$AI=\left(\frac{r(x^0,p)-\sum_{r=1}^{s}p_r y_r^0}{s\cdot\ \min\{p_1y_1^0,\ldots,p_sy_s^0\}}\right)-RI_o(x^0,y^0). \tag{24}$$

Therefore, NRI can be easily decomposed into technical and allocative components. We also note that NRI is always greater than or equal to zero with nil inefficiency signaled when it takes a value of zero. The same interpretations can be given to the technical and allocative subcomponents. Indeed, NRI satisfies the condition that it is zero if and only if the DMU being assessed achieves maximum revenue at market prices, an interesting property called indication in the literature (Portela and Thanassoulis, 2007).

Finally, in the next section, we illustrate our method by means of a numerical example.

4. Numerical example

This section includes a numerical illustration on the use of the new methodology proposed in this paper. In particular, we use a database with 11 DMUs that use one input to produce two outputs (see Table 1). Seeking simplicity, we have considered that all DMUs have the same level of input, equal to one. Additionally, we have assumed that some DMUs produce zero for one of their outputs. We have avoided the situation where all outputs are zero for the same DMU since this seems an unusual case. The DEA technology estimated from the considered data sample is graphically illustrated in Fig. 2. As can be seen, DMUs C, D, E and F belong to the strong efficient frontier. Additionally, we assume that market output prices are $p_1=1$ and $p_2=4$. In this way, maximal revenue is achieved at point C.

Table 1 records, for all the DMUs, the results obtained when we apply the approach presented in this paper. To be precise, for each of these units we have reported the value of its input and outputs in the first three columns. The fourth column is the Russell output measure of technical efficiency, i.e., the optimal value of model (8), denoted as $R_o(x^0,y^0)$. The projection point generated by (8) is shown in the fifth column. The last three columns correspond to the normalized revenue inefficiency as in (22), the value of the Russell output measure of inefficiency obtained from (10) and the allocative inefficiency component calculated by means of (24).

It is worth mentioning that the Russell measures are well-defined for evaluating vectors of strictly positive outputs. Otherwise, several interesting properties of these measures can fail. As an example, let us focus on the analysis on DMU A. If we use model (8) to determine the Russell output measure of technical efficiency for this unit, we obtain that the objective function would tend to $+\infty$. The solution that we consider suitable for this situation consists of deleting the terms related to outputs with zero from the objective functions in (3) and (8), restricting the minimum in (2) over the terms with outputs strictly positive, and changing s by the number of outputs strictly positive for each assessed DMU. However, in this case, property (ii) in Proposition 2 is no longer met. To illustrate this point, note that $R_o(x^A,y^A)=1$. However, DMU A is a unit that is dominated in the sense of Pareto by DMU B.

Nevertheless, despite the fact that for DMUs A, H, and I, the use of the Russell measure cannot be completely justified since Pareto–Koopmans efficiency is not always related to a value of one for this measure, our results of duality remain true as we pointed out in footnote 3. Note how for these three units expression (22) holds, in this case with equality. The equality is due to the fact that, for example for I, economic inefficiency, technical inefficiency and allocative inefficiency are measured exclusively with respect to the second output. For this particular unit, the only source of inefficiency detected is technical, as a consequence of moving from I to A, and, additionally, there is not allocative inefficiency (AI=0) and, therefore, NRI=TI. Anyway, we recognize that the problem of the traditional Russell measures for dealing with zero values in inputs and outputs is a limitation that is inherited by our approach. Nevertheless, this feature has not prevented Russell measures from recently being successfully applied to different contexts (see, for example, Lansink and Ondersteijn, 2006; and Mahlberg and Sahoo, 2011) under the assumption of working with nonzero outputs.

Regarding the analysis of the results for DMUs with strictly positive outputs, we want to point out two things. First, inequality (22) is satisfied for all these units. Second, C is the only revenue efficient unit (NRI=0), which implies that it is also technically and allocatively efficient, whereas D, E, and F are technically efficient (TI=0) but revenue inefficient (NRI > 0), with this inefficiency being purely allocative.

Finally, let us show how an additional property of the original Russell measure is also inherited in some sense by the proposed approach. We are referring to the possibility of multiple solutions in the case of the application of model (8), the non-uniqueness property. We graphically illustrate this situation in Fig. 2 and DMU J. For this unit, there are two optimal projections that generate the same value for the Russell output measure: units D and E. However, NRI and TI in (22) and AI in (24) have the same value regardless of the projection selected on the strong efficient frontier.

5. Conclusions

In this paper, we have introduced a new way to measure and decompose revenue inefficiency. Our approach was based on the

[5] In contrast to what is usually believed, the Russell output measure of inefficiency $RI_o(x^0,y^0)$ is additive in nature. If we apply the change of variables $\phi_r=(y_r^0+s_r^+/y_r^0)=1+(s_r^+/y_r^0)$, $r=1,\ldots,s$, we get that program (8) is equivalent to $1+\underbrace{\max_{s^+\geq 0,\lambda\geq 0}\left\{\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_r^0}:\sum_{j=1}^{n}\lambda_j x_i^j\leq x_i^0,\forall i,-\sum_{j=1}^{n}\lambda_j y_r^j+s_r^+\leq -y_r^0,\forall r,\sum_{j=1}^{n}\lambda_j=1\right\}}_{v^*(x^0,y^0)}$. In this way, by (10), we have that $RI_o(x^0,y^0)=v^*(x^0,y^0)$ and, therefore, the Russell output measure of inefficiency is equivalent to an additive-type measure. The additive nature of the Russell measures has already been noted by Färe et al. (2007) and more extensively and formally studied in Fukuyama and Weber (2009).

Table 1
Results corresponding to the numerical example.

DMU	Input	Output 1	Output 2	$R_o(x^0, y^0)$	Projection	NRI	TI	AI
A	1	0	4	1	A	0	0	0
B	1	1	4	1.5	C	0.5	0.5	0
C	1	2	4	1	C	0	0	0
D	1	3	3.5	1	D	0.167	0	0.167
E	1	3.5	3	1	E	0.357	0	0.357
F	1	4	2	1	F	0.75	0	0.75
G	1	4	1	1.5	F	1.25	0.5	0.75
H	1	4	0	1	H	0	0	0
I	1	0	2	2	A	1	1	0
J	1	2	2	1.625	D	2	0.625	1.375
K	1	3	1	2.25	D	1.833	1.25	0.583

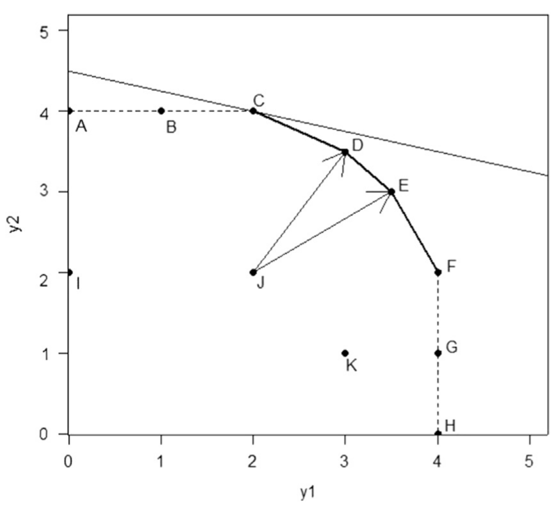

Fig. 2. Illustration of the numerical example.

output-oriented Russell measure. This fact allowed us to assure that the technical efficiency component accounts for all the sources of inefficiency on the output side, including the output slacks. Although the input and output oriented versions of the Russell measure are well known in the literature, as far as we are aware, this is the first approach employing these technical measures that achieves decomposing overall inefficiency.

All our results were derived from a new Fenchel–Mahler inequality resorting to the application of the theory of convex conjugates. We particularly demonstrated the existence of a new dual relationship between the revenue function and the Russell output measure, in contrast to what is believed by some authors.

Regarding the implications of the results of this paper, we note two points. From a theoretical point of view, we have been able to establish a natural dual relationship between the revenue function and the original version of the Russell output measure; in spite of the previous multiple unsuccessful attempts to provide such duality results (see Kopp, 1981, Russell, 1985, Färe et al., 1985 and Färe et al., 2007). From a practical point of view, we also provide a way of decomposing revenue inefficiency into allocative inefficiency and a component that measures technical inefficiency in the sense of Pareto, taking into account all the existing sources of technical inefficiencies (slacks). This feature contrasts with the usual approaches for decomposing revenue inefficiency, such as those based on Shephard's output distance function (Shephard, 1953) and the directional output distance function (Chambers et al., 1996, 1998), which ignore the possible existence of slacks related to the projected points on the production frontier.

Finally, it is worth mentioning that all the results derived in this paper for the output-oriented Russell measure and the revenue function can also be obtained for the input-oriented version of the Russell measure and the cost function.

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